Comput. Math. Applic. Vol. 15, No. 4, pp. 291–298, 1988
Printed in Great Britain.

0097-4943/88 \$3.00 + 0.00

NODE-GRACEFUL GRAPHS

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Abstract—A disconnected forest $F = (V, E)$ with $|V| = p$ and $|E| = q$ cannot be graceful as there is no possible numbering of V with distinct integers from the set $\{0, \ldots, q\}$. However, the augmented set $\{0, \ldots, p-1\}$ has just enough numbers for V and suggests the concept of a node-graceful graph. On the other hand, some problems arising from radar and sonar sequences of distinct frequencies in consecutive time slots can be regarded as the two-dimensional analog of the one-dimensional "ruler problems". These two-dimensional synchronization patterns are formulated in terms of node-graceful graphs. It is shown that the matching graph nK_2 is node-graceful if and only if there exists an (s, n)-Skolem sequence with $s = 2$. Other results are obtained and the current state of knowledge is summarized.

1. INTRODUCTION

Recently much interest has developed in studying various "graph numberings". Typically, the nodes of an undirected graph are assigned values, which in turn determine values upon each edge as a function of the two values on the nodes of the edge. A wide variety of these numberings has been studied both for their intrinsic combinatorial interest and for their application to an expanding range of domains. Some of these can be found in [1, 2, 3] which cite applications to radar pulse codes, X-ray crystallography, circuit layout design and missile guidance, among others. In addition, applications have been studied for communication loop addressing [4], radioastronomy [5], coding theory [6] and broadcast frequency assignments [7, 8].

The problems mentioned above include graceful graphs, harmonious graphs, elegant graphs and the bandwidth problem. More recently, graceful numberings on directed graphs are also studied and applications have been established to algebraic systems, generalized complete mappings, network addressing problems, and the n-queen problem [9, 10, 11].

Golomb and Taylor [12] formulated, in terms of square or rectangular arrays of dots with appropriate constraints on the two-dimensional correlation function, a number of closely related problems corresponding to specific assumptions about the type of time-frequency sequence which may be useful in a particular application. These problems may be regarded as the two-dimensional analogue of the one-dimensional "ruler problems" described at length by Bloom and Golomb [2], which have application to one-dimensional synchronization and alignment problems, and to radar or sonar situations where the Doppler shift can be neglected.

We now introduce the concepts of graceful node access, graceful edge access, node-gracefulness and we study graphs, especially forests, which are node-graceful. This problem may be considered as a generalization of the graceful graph problem. We also show that two-dimensional radar sequences can be constructed by using certain node-graceful graphs.

It is shown that the disjoint union of n copies of K_2 is node-graceful if and only if there exists an (s, n)-Skolem sequence with $s = 2$. It follows [13, 14] that nK_2 is node-graceful if and only if $n \equiv 0$ or 1 (mod 4). Another infinite family of node-graceful forests is given by $2K_{1,2n}$, a pair of stars of equal even size. The current state of knowledge concerning node-graceful graphs is summarized.

2. NODE-GRACEFUL GRAPHS

Let G be an undirected graph with no loops and no multiple edges. Assume G has p nodes and q edges. It is not necessary for G to be connected. We follow in general the notation and terminology of [15].

A *graceful numbering* of G is a mapping α from the node set $V(G)$ into $\{0, 1, 2, \ldots, q\}$ in such a way that the set of edge numbers equals $\{1, 2, \ldots, q\}$ when edge ab is assigned the number

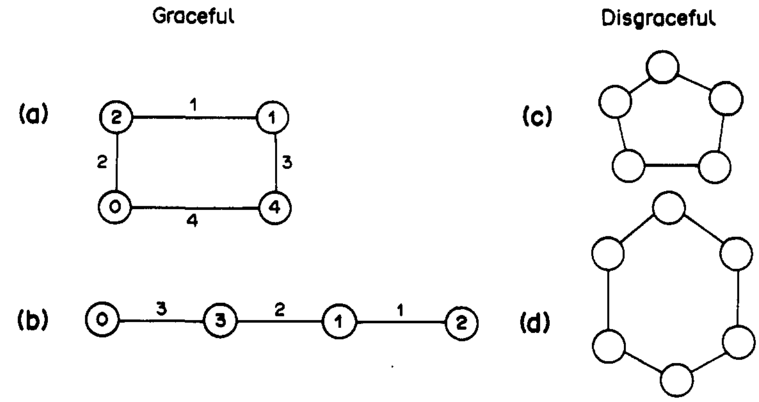


Fig. 1. Examples of graceful and disgraceful connected graphs.

$\alpha(ab) = |\alpha(a) - \alpha(b)|$. A graph G is said to be *graceful* if it admits a graceful numbering. A graph which is not graceful is called *disgraceful.*

Examples of connected undirected graphs are shown in Fig. 1. The two graphs on the left are graceful with given graceful numberings and the two on the right are not graceful. The formidable problem of characterizing graceful graphs still remains unsolved. In fact, despite considerable effort, it is still not yet known whether all trees are graceful. For a recent survey on graceful graphs, see [16]. Earlier surveys of graceful graphs and their applications are given in [1, 2, 3].

Since the nodes of a graceful graph are numbered using the set $\{0, 1, 2, 3, \ldots, q)$, we have $p \leqslant q + 1$. A *forest* is a graph with no cycles. A *tree* is a connected forest. In a tree, we have $p = q + 1$. Hence, there exist graceful and disgraceful graphs with more than one component as shown in Fig. 2.

Graceful graph problems have also been studied for disconnected graphs but not as extensively as for connected graphs. This is probably due to lack of motivation and applications. Recently, however, we have found that some classes of graphs, though not graceful, are very close to being graceful and have applications to the construction of two-dimensional radar sequences which generalizes the one-dimensional ruler problem studies by Bloom and Golomb [2].

Some disgraceful graphs have a "graceful deficiency" and others do not. For example, those cycles [as in Figs 1(c) and (d)] which are not graceful can be given a node numbering which makes the edges have numbers $1, 2, \ldots, q$ as required in a graceful numbering, but the numbers on the nodes need to be greater than those permitted in a graceful numbering.

Let G be a (p, q) graph and m the minimum integer which enables the set $\{0, 1, \ldots, m\}$ to number the nodes of G so that the edge ab is numbered as $|a - b|$ and the edges are distinctly numbered $\{1, 2, \ldots, q\}$. The *graceful node access* (gna) is defined to be gna $= m - (p - 1)$ and the *graceful edge access* (gea) is defined to be gea $= m - q$.

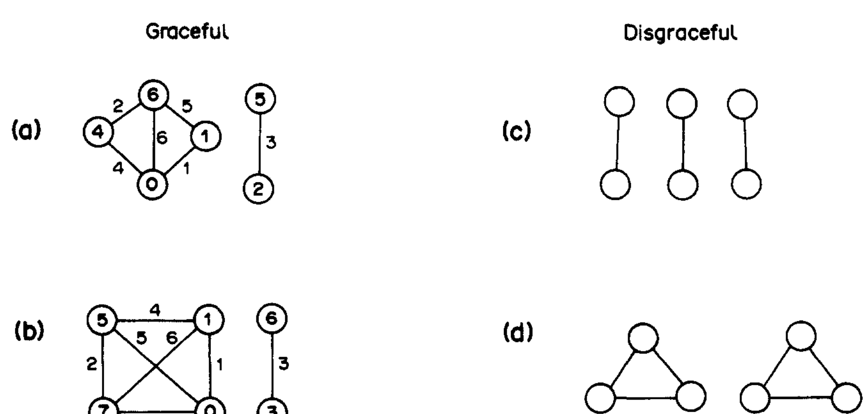


Fig. 2. Examples graceful and disgraceful disconnected graphs.

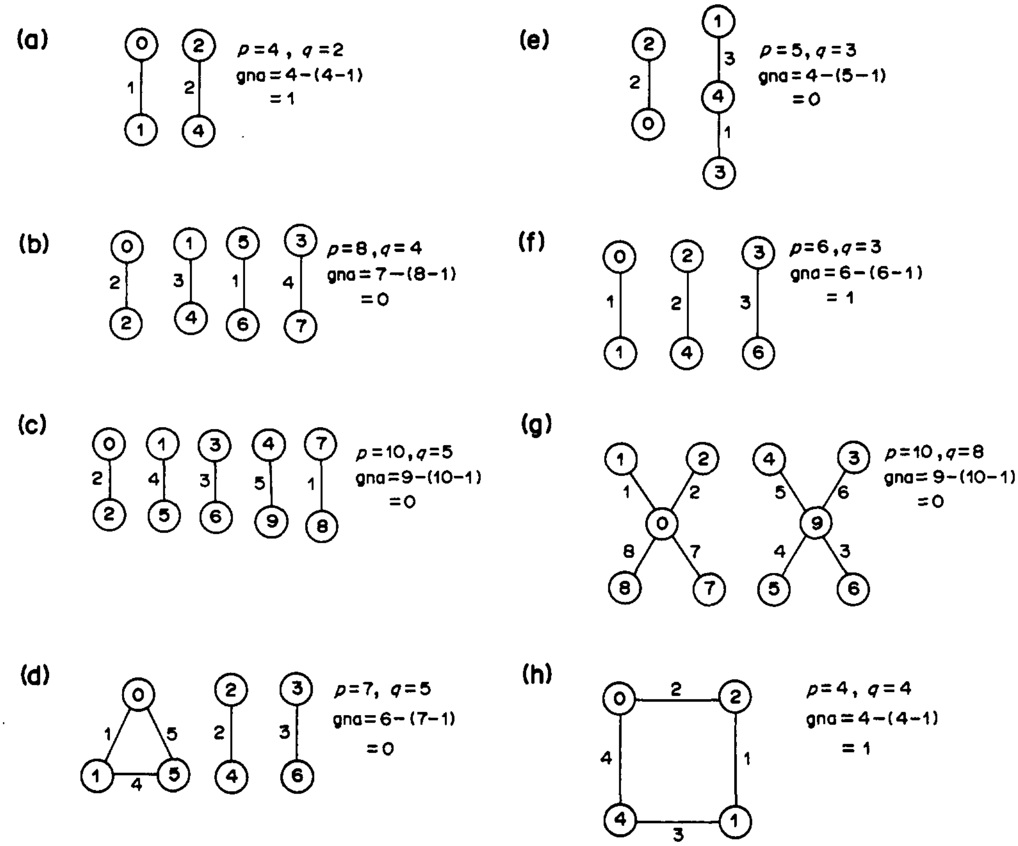


Fig. 3. Some graphs with their graceful node access.

As noted before, not all graphs have a number m. The complete graph K_n, $n > 4$, does not possess such a number m. Hence it does not have a gna or gea. It is interesting to note that gea $\leqslant 0$ for a graceful graph. However, graceful node access gna $= 0$ does not imply that a graph is graceful, see Fig. 3(g). On the other hand, it is necessary for a graceful graph to have gna $= 0$, see Fig. 3(h). In fact, we are more interested in which disgraceful graphs have graceful node access zero.

A (p, q) graph G is said to be *node-graceful* if G has graceful node access gna $= 0$ and it admits a node-graceful numbering. In other words, we can use $\{0, 1, 2, \ldots, p-1\}$ to number the nodes to obtain an edge numbering by $\{1, 2, \ldots, q\}$. Figure 3 shows some disgraceful graphs with various graceful node access. All graphs in Fig. 3 are disgraceful except (h). Graceful edge accesses of these graphs are not shown since these numbers can be calculated from graceful node access using the definition, that is,

$$\text{gea} = m - q = (\text{gna} + p - 1) - q = \text{gna} + p - q - 1.$$

All the node-graceful graphs of Fig. 3 are shown with their node-graceful numberings.

Although the study of graceful access can be regarded as a generalization of the graceful graph problem, the concept is rather distinct. There are node-graceful graphs which are not graceful. There are also graceful graphs which are not node-graceful as shown in Fig. 3(h). The latter case arises when a graph has $p = q$. Hence it is clear that a disgraceful cycle can not be node-graceful.

Lemma 2.1

If a cycle C_n is not graceful, then gna $(C_n) \geqslant 2$.

Proof. Since C_n is not graceful, the maximum number m required to number the nodes satifies $m > q = n$. Hence the graceful node access

$$\text{gna} = m - (p-1) > n - (n-1) = 1.$$

that is, gna $(C_n) \geqslant 2$. □

As a corollary to Lemma 2.1, a disgraceful cycle cannot be node-graceful. Moreover, Lemma 2.1 is also true for a disgraceful graph with $p = q$. We now develop a necessary and sufficient condition for nK_2, the disjoint union of n copies of K_2, to be node-graceful. To do this, we require the notion of a Skolem sequence.

The Skolem problem is the determination of the values of integers s and n greater than 1 for which there exists a sequence of length sn such that each of the integers $1, 2, \ldots, n$ occurs exactly s times and successive occurences of i are separated by exactly $i - 1$ terms of the sequence, $1 \leqslant i \leqslant n$. Such a sequence is called an *(s, n)-Skolem sequence*. For example,

(1) 1 1 2 8 2 3 7 5 3 6 4 8 5 7 4 6,

(2) 6 3 1 1 3 8 6 7 4 2 5 2 4 8 7 5,

(3) 8 6 4 2 7 2 4 6 8 3 5 7 3 1 1 5

are three (2, 8)-Skolem sequences, see [17, 18], while

(4) 1 1 1 2 10 2 7 2 9 3 6 8 3 7 10 3 6 9 5 8 7 4 6 5 10 4 9 8 5 4

is a (3,10)-Skolem sequence [18].

These sequences were originally studied by Skolem [13] in connection with the construction of cyclic Steiner triple systems. Closely related are the (s,n)-Langford sequences which differ from Skolem sequences only in that successive occurences of i are separated by exactly i terms of the sequence, $1 \leqslant i \leqslant n$, see [17, 18].

It is obvious that every (s,n)-Langford sequence can be converted to an $(s,n + 1)$-Skolem sequence by either prefixing or suffixing a string of s zeros. For clarity, we illustrate with one example. The following sequence

(5) 1 7 1 2 6 4 2 5 3 7 4 6 3 5

is a (2, 7)-Langford sequence. By prefixing a string of two zeros to (5), we have the sequence

(6) 0 0 1 7 1 2 6 4 2 5 3 7 4 6 3 5.

After adding 1 to each term of series (6), the resulting sequence is certainly a (2, 8)-Skolem sequence. However, the converse of the process is not true. The two examples (2) and (3) of (2, 8)-Skolem sequences can not be converted to (2, 7)-Langford sequences. The survey paper by Roselle [18] presents an account of the history of these problems.

Given a $(2, n)$-Skolem sequence $a_1, a_2, \ldots, a_{2n}$, we can construct a numbering of nk_2 as follows: number the edges of nK_2 as $1, 2, 3, \ldots, n$ and number the two nodes of the edge i as j and k such that $a_j = a_k = i$. Since successive occurrences of i in the $(2,n)$-Skolem sequence are separated by exactly $i = 1$ terms of the sequence, $|k - j| = (i - 1) + 1 = i$. Hence the numbering is node-graceful.

Theorem 2.2

The graph nK_2 is node-graceful if and only if $n \equiv 0$ or 1 (mod 4).

Proof. It was shown independently by Skolem [13] and Marsh [14] that a $(2, n)$-Skolem sequence exists if and only if $n \equiv 0$ or 1 (mod 4). □

For brevity we write S_n for the star $K_{1,n}$. We define a *galaxy* to be a disjoint union of stars, written $S_{k_1} \cup S_{k_2} \cup \cdots \cup S_{k_t}$. If each component is the same S_k, then we write tS_k. Note that $2P_3 = 2S_2$ is node-graceful with numbering [1, 0, 4], [2, 5, 3]. Figure 3(g) shows that $2S_4$ is node-graceful.

Theorem 2.3

For all positive k, the galaxy $2S_{2k}$ is node-graceful.

Proof. The galaxy $2S_{2k}$ has $p = 2(2k + 1) = 4k + 2$ and $q = 2(2k) = 4k$. If suffices to construct a node-graceful numbering.

Let $(c; l_1, l_2, \ldots, l_k)$ be used to number the nodes of S_k with the center node numbered c. There are two components of the galaxy $2S_{2k}$. We number the nodes of these two components as:

$$(0; 1, 2, \ldots, k, 3k + 1, 3k + 2, \ldots, 4k)$$

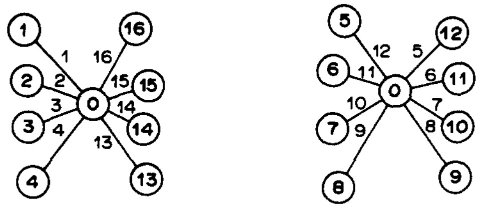


Fig. 4. A node-graceful numbering for $2S_8$, $p = 18$, $q = 16$.

and

$$(4k + 1; k + 1, k + 2, \ldots, 2k, 2k + 1, 2k + 2, \ldots, 3k).$$

The first component has edge numbers $\{1, 2, 3, \ldots, k, \ 3k + 1, \ 3k + 2, \ldots, 4k\}$ while the second component has $\{3k, \ 3k - 1, \ldots, 2k + 1, \ 2k, \ 2k - 1, \ldots, k + 1\}$. These two sets comprise $\{1, 2, 3, \ldots, k, k + 1, \ldots, 3k, \ 3k + 1, \ldots, 4k\}$ which is exactly the same as $\{1, 2, \ldots, q\}$. Hence the numbering is node-graceful. $\square$

As an example of Theorem 2.3, we take $k = 4$, so that $p = 18$, $q = 16$ and $4k + 1 = 17$. We then have the following node-graceful numbering for $2S_8$; see Fig. 4:

$$2S_8\colon (0; \ 1, 2, 3, 4, 13, 14, 15, 16)$$

and

$$(17; 5, 6, 7, 8, 9, 10, 11, 12).$$

3. RADAR SEQUENCES

It is well-known that the one-dimensional "ruler problem" can be modeled by graceful graphs. As an example, the ruler of length 6 as in Fig. 5(a) only requires 4 marks in order to measure all the lengths from 1 to 6. This set of marks is equivalent to a graceful numbering of the complete graph of 4 nodes as shown in Fig. 5(b).

These problems are described at great length by Bloom and Golomb [2]. Recently, Golomb and Taylor [12] have formulated a number of closely related combinatorial problems corresponding to specific assumptions about the type of time–frequency sequence in terms of square or rectangular arrays of dots with appropriate constraints on the two-dimensional correlation function.

The central patterns considered by Golomb and Taylor have the property that in any position reachable by horizontal and vertical noncyclic shifting other than the original position, the pattern will overlap with the original in at most one dot location. Figure 6 shows two examples from [12] in which the number of dots is maximized for a 3×3 and a 5×5 array.

A special kind of two-dimensional agreement pattern (mentioned above) was observed by H. Greenberger in which the practical application to Doppler sonar or radar does not require the restriction to one dot per row. In a pattern which has only the restriction of one dot per column, it can be read like music notation giving a sequence of tones, but with only one tone at each beat. When the "tones" return after being reflected from a moving target, horizontal shift will correspond to the Doppler Effect. These patterns are called *sonar sequences*. If the Doppler measurement is not required, these patterns are called *radar sequences*. Examples of a sonar sequence and a radar sequence are given in Fig. 7.

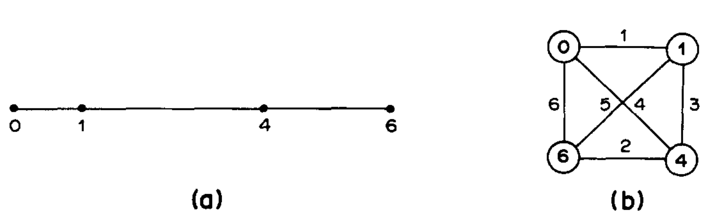


Fig. 5. A ruler of length 6 and its corresponding graph.

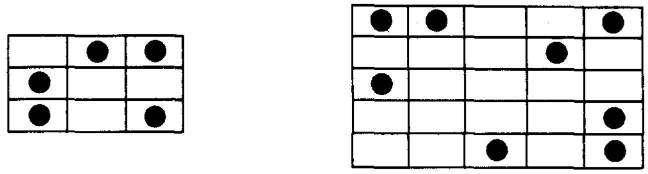
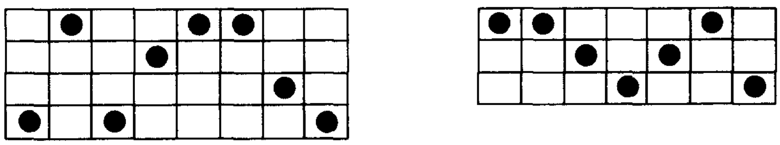


Fig. 6. Optimum 3×3 and 5×5 arrays.



(a) (b)

Fig. 7. A 4×8 sonar sequence and 3×7 radar sequence.

If we number the columns in the two-dimensional array in Fig. 7(b) which represents a 3×7 radar sequence as $0, 1, 2, 3, 4, 5, 6$, the dots in each row represent the positions with respect to each column number. If each row of k dots is represented by a complete graph of k nodes, then the radar sequence in Fig. 7(b) gives a graph numbering of $K_3 \cup 2K_2$ in which the nodes are numbered by their corresponding column number determined by the location of the dots. We define the *row difference* to be the position difference between dots in the same row. Since after any horizontal shifting the pattern will overlap with the original in at most one dot location, the row difference should be distinct within a row or across all rows. The row differences correspond to the edge numberings of $K_3 \cup 2K_2$. It turns out that the resulting graph numbering is node-graceful as shown in Fig. 3(d). The graph numbering corresponding to a radar sequence as discussed has distinct edge numbers. Hence it is not necessary that it be node-graceful. This can be illustrated in Fig. 8 where the radar sequence is taken from Fig. 7(a) and the corresponding graph has a numbering from $\{0, 1, 2, 3, 4, 5, 6, 7\}$ with all edge numbers distinct. But the edge numbers do not constitute the numbers from 1 to $q = 6$. However, if we start with a specific node-graceful graph with a node-graceful numbering, a radar sequence can be constructed.

Theorem 3.1

If the disjoint union G of t complete graphs

$$\bigcup_{i=1}^{t} K_{n_i}$$

is node-graceful, then there exists a radar sequence $R(G)$ of t rows and c columns where

$$c = \sum_{i=1}^{t} n_i.$$

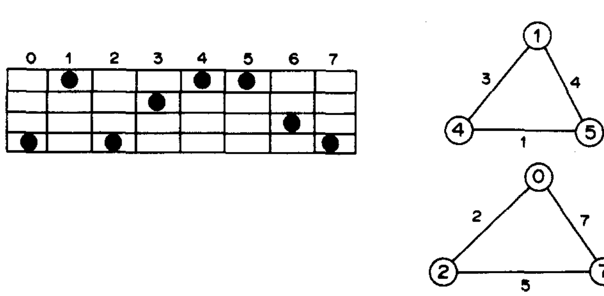


Fig. 8. A 4×8 radar sequence with columns numbered and its corresponding graph fully labeled.

Proof. Let the two-dimensional array which represents a radar sequence be numbered as $1, 2, \ldots, t$ for rows as $0, 1, 2, \ldots, c-1$ for columns. Each complete graph K_{n_i} in G corresponds to the ith row in the array. We construct the array in the following way. There exists a dot in the (i, j) entry if and only if the number j appears in the node-graceful numbering of K_{n_i}. It remains to show that the constructed array is a radar sequence.

The graph G has $\Sigma n_i = c$ nodes. Since the node-graceful numbering uses distinct numbers in $\{0, 1, 2, \ldots, c-1\}$ on the nodes of the graph G, each column in the array can only have one dot. Moreover, since the edge numbers are all distinct in $\{1, 2, 3, \ldots, q)$ all the row differences are distinct within one row, i.e., within the same component in the graph, or across all row, i.e., across all components. Therefore the sequence $R(G)$ constructed from G is a radar sequence. $\square$

It is clear that the row differences in any radar sequence constructed from Theorem 3.1 are all distinct and range over $\{1, 2, 3, \ldots, q\}$, where q is the number of edges in the corresponding graph $\cup K_{n_i}$. If we examine this number closely, we realize that q is also the number of possible row differences in the radar sequence. This suggests the following result, which is the converse of Theorem 3.1 and can be proved accordingly.

Theorem 3.2

Let R be a radar sequence with t rows and c columns when represented as a two-dimensional array. Let q be the number of possible row differences in R. We number the columns of R by $0, 1, 2, \ldots, c-1$ and compute the row difference between two dots at column position i and j as $|i-j|$. If all the possible row differences constitute the set $\{1, 2, \ldots, q\}$, then there exists a node-graceful numbering of the graph

$$G(R) = \bigcup_{i=1}^{t} K_{n_i},$$

which is the disjoint union of t complete graphs, with q edges and c nodes,

$$c = \sum_{i=1}^{t} n_i. \qquad \square$$

4. CONCLUDING REMARKS

The current study of graph numbering problems generates many interesting problems. First, we state some general open questions:

(*a*). Which linear forests are node-graceful? We know that those in Figs 3(b), (c) and (e) are. As mentioned above, $2P_3$ is also node-graceful. So are $P_2 \cup P_n$ for $n = 3, 4, 5$ with node-graceful numberings [02] [143], [03] [5124], and [04] [61253], respectively.

(*b*). Which galaxies are node-graceful? Those in Fig. 3(g) and Theorem 2.4 are. It is stated in [19] that Lee and Wise proved $S_{k_1} \cup S_{k_2}$ is node-graceful if and only if $k_1 k_2$ is even, and $S_{k_1} \cup S_{k_2} \cup S_{k_3}$ is node-graceful if and only if $k_1 k_2 k_3$ is even. The general problem for $t \geqslant 4$ stars in a galaxy is still open.

(*c*). Characterize those graphs with graceful node access, gna $= c$ for each positive integer $c = 1, 2, 3, \ldots$.

In a node-graceful numbering of a node-graceful graph, the nodes are numbered $0, 1, 2, \ldots, p-1$ and the edges $1, 2, 3, \ldots, q$. It is easy to see that $p - 1 \geqslant q$ since the only way to get an edge numbered q is with its two nodes numbered as 0 and $p - 1 = q$. Hence we have $p \geqslant q + 1$. On the other hand, if a graph G is graceful, then $q + 1 \geqslant p$. Therefore, a necessary condition for a graph G to be both graceful and node-graceful is that $p = q + 1$. If G is connected, then G must be a tree. Thus when $p = q + 1$, the concepts of a node-graceful graph and a graceful graph coincide. In this case, we call the graph *critical* and the numbering is called a *critical numbering*. It is not known whether all trees are critical. Here we ask the following question:

(*d*). Characterize critical graphs which are not connected. None of the graphs in Fig. 3 are critical. A critical numbering for $C_5 \cup P_2$ is given by Lee and Shee [19].

The necessary condition $p \geqslant q + 1$ for a graph to be node-graceful is very helpful in determining when a graph is not node-graceful. Let tK_n, $n \geqslant 2$, be the disjoint union of t complete graphs K_n.

Then the order is $p(tK_n) = tn$ and the size is $q(tK_n) = tn(n-1)/2$. Hence for $n \geqslant 3$, obviously $q + 1 > p$. This implies that tK_n is not node-graceful when $n \geqslant 3$. Therefore the only node-graceful graphs in which all the components are complete graphs of the same order have the form nK_2. If we don't require each component to have the same order, the situation is quite different. As mentioned above, $K_3 \cup 2K_2$ is node-graceful although it is not critical.

In Theorems 3.1 and 3.2, those radar sequences where q is the number of possible row differences constructed from node-graceful graphs of the form

$$\bigcup_{i=1}^{t} K_{n_i}$$

have the special property that the row differences range over $\{1, 2, 3, \ldots, q\}$. We call these radar sequences *graceful radar sequences*.

(e). Characterize and construct graceful radar sequences.

Finally, we note that if the graph G with p nodes and q edges is critical, then ${}_2K_p$, the multigraph obtained from the complete graph K_p by doubling each edge, can be decomposed into the edge disjoint union of p copies of G. We illustrate this by an example mentioned previously. Since $C_5 \cup P_2$ is critical with the critical numbering (0 2 5 1 6) [3 4], we write $G = G_0 = (0\ 2\ 5\ 1\ 6)\ [3\ 4]$. Let G_i be isomorphic to G with the numbering $(0+i\ 2+i\ 5+i\ 1+i\ 6+i)\ [3+i\ 4+i]$. It is easily verified that

$${}_2K_7 = \bigcup_{i=0}^{6} G_i.$$

This decomposition is cyclic since each G_i is a translate of G_0. The study of critical graphs provides new constructions for cyclic decompositions of graphs.

REFERENCES

1. G. S. Bloom, Numbered undirected graphs and their uses: A survey of a unifying scientific and engineering concept and its use in developing a theory of non-redundant homometric sets relating to some ambiguities in X-ray diffraction analysis. Ph.D. Dissertation, University of Southern California (1975).
2. G. S. Bloom and S. W. Golomb, Applications of numbered undirected graphs. *Proc. IEEE* **65**, 562–570. (1977).
3. G. S. Bloom and S. W. Golomb, numbered complete graphs, unusual rulers, and assorted applications. *Springer Lecture Notes Math.* **642**, 53–65 (1978).
4. R. L. Graham and H. O. Pollack, On the addressing problem for loop switching. *Bell Syst. Tech. J.* **50**, 2495–2519 (1971).
5. J. C. Bermond, Graceful graphs, radio antennae, and French windmills. In *Graph Theory and Combinatorics*, pp. 18–37. Pitman, London (1979).
6. R. L. Graham and N. J. A. Sloane, On constant weight code and harmonious graphs. *Congressus Numerantium* **26**, 25–40 (1980).
7. M. B. Cozzins and F. S. Roberts, T-colorings of graphs and the channel assignment problem. *Congressus Numerantium* **35**, 191–208. (1982).
8. J. H. Radinowitz and V. K. Proul, An asymptotic approach to the channel assignment problem (in press).
9. G. S. Bloom and D. F. Hsu, On graceful graphs and a problem in network addressing. *Congressus Numerantium* **35**, 91–103 (1982).
10. G. S. Bloom and D. F. Hsu, Graceful directed graphs, *SIAM Jl Algebraic Discrete Methods* **6**, 519–536 (1985).
11. G. S. Bloom and D. F. Hsu, On graceful directed graphs that are computational models of some algebraic systems. In *Graph Theory and Its Applications to Algorithms and Computer Science*, pp. 89–102. Wiley, New York (1985).
12. S. W. Golomb and H. Taylor, Two-dimensional synchronization patterns for minimum ambiguity. *IEEE Trans. Inf. Theory* **28**, 600–604 (1982).
13. T. Skolem, On certain distributions of integers in pairs with given differences. *Math. Scand.* **5**, 57–68 (1957).
14. D. C. B. Marsh, Solution to Problem E1845. *Am. math. Mon.* **74**, 591–592 (1967).
15. F. Harary, *Graph Theory*. Addison–Wesley, Reading (1969).
16. G. S. Bloom, A chronology of the Ringel–Kotzig conjecture and the continuing quest to call all trees graceful. *Ann. N.Y. Acad. Sci.* **328**, 31–51 (1979).
17. J. F. Dillon, The generalized Langfor-Skolem problem. *Congressus Numerantium*, 237–247 (1973).
18. D. P. Roselle, Distribution of integers into s-tuples with given differences. *Proc. Manitoba Conf. on Numerical Math.*, Winnipeg, Manitoba, pp. 31–42 (1971).
19. S-M Lee and S. C. Shee, On Skolem-graceful graphs (in press).